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Int. J. Heat Mass Transfer. Vol. 28, No. 3, pp. 717–720, 1985
Printed in Great Britain
0017-9310/85 $3.00 + 0.00
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A Monte Carlo procedure for straight convecting boundaries

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(*Received* 13 *October* 1983 *and in revised form* 2 *July* 1984)

NOMENCLATURE

a coefficient
A upper limit of integration, $A = \int_{-\pi/2}^{\pi/2} P(z)\,dz$
b coefficient
B coefficient,
$$B = \int_{-\pi/2}^{\pi/2} [1-(B_i/3)\cos(3z)]P(z)\,dz$$
B_i Biot number, $B_i = hR/k$
C constant
d_{ave} average distance reflected into interior perpendicular to boundary
e natural number, e = 2.71828...
E sum of squared errors, see equation (16)
f_n function of z, $f_n(z)$
$$= \cos(nz)+B_i\cos[(n+1)z]/(n+1),\ n \text{ even}$$
$$= \sin(nz)+B_i\sin[(n+1)z]/(n+1),\ n \text{ odd}$$
$\overline{f_n^2}$ function of $B_i, \overline{f_n^2}$
$$= \pi+4B_i+\pi B_i^2/2,\ n = 0$$
$$= \pi/2+4B_i/(2n+1)+\pi B_i^2/[2(n+1)^2],\ n \neq 0$$
F cumulative distribution function,
$$F = \int_{-\pi/2}^{z} P(z')\,dz'/A$$
G function of x, see equation (10)
h heat transfer coefficient
k thermal conductivity
L distance reflected in one-dimensional approximation
L_1, L_2 differential operators, see equation (10)
N random number
P probability density function, see equation (17)
q negative derivative of T with respect to z, $q = -\partial T/\partial z$
q'' heat release rate per unit area
q_0 function of x, see equation (13)
Q dimensionless heat release rate parameter, $Q = q''R^2/4k$
r distance from the vertex
R straight side length
T temperature
$T_{boundary}$ boundary temperature
$T_{interior}$ temperature at distance L into interior
T_0 function of x, see equation (12)
T_p peripheral temperature
T_v vertex temperature, $T_v = T(x \to \infty)$
T_∞ environmental temperature
x dimensionless coordinate, $x = \ln(R/r)$
y temperature parameter, $y = T_p(z)-T_\infty+Q[1-(B_i/3)\cos(3z)]$
z dimensionless coordinate, $z = \theta-\theta_m/2$.

Greek symbols
α differential operator, $\alpha = \partial(\ )/\partial x$
θ angle from the straight side
θ_m included angle
ζ differential operator, $\zeta = \partial(\ )/\partial z$
π natural number, $\pi = 3.14159\ldots$.

Subscripts
m index
n index.

INTRODUCTION

THE FLOATING random walk Monte Carlo method for two-dimensional steady conduction requires a procedure to determine the next move of a walker situated on a boundary that convectively exchanges heat with an environment at known temperature across a known heat transfer coefficient. Previously [1], a one-dimensional finite difference approximation to the convective boundary condition has been employed in the direction perpendicular to the boundary as

$$T_{boundary} = [T_{interior}+(hL/k)T_\infty]/[1+hL/k].$$

Such an approximation requires the reflective move back into the solid to be small to preserve accuracy and to be perpendicular to the boundary. A two-dimensional approximation is subject to the same difficulties.

An amelioration will be devised from a solution for the vertex temperature of a sector of a circle experiencing convection along its straight sides and with a specified temperature along the circular arc, the vertex being on the convective boundary.

VERTEX TEMPERATURE IN A SECTOR OF A CIRCLE

The temperature distribution in the sector of a circle shown in Fig. 1 is described by

$$\partial^2 T/\partial x^2 + \partial^2 T/\partial z^2 + 4Q\,e^{-2x} = 0 \quad (1)$$

$$T(\infty, z)\ \text{finite} \quad (2)$$

$$\partial T(x, -\theta_m/2)\partial z = B_i\,e^{-x}[T(x, -\theta_m/2) - T_\infty] \quad (3)$$

$$\partial T(x, \theta_m/2)/\partial z = -B_i\,e^{-x}[T(x, \theta_m/2) - T_\infty] \quad (4)$$

$$T(0, z) = T_p(z) \quad (5)$$

for constant thermal conductivity and heat release rate.

The mixed method of Rao [2] is applied to generate the solution. It begins by expanding $T(x, z)$ in a MacLaurin series as

$$T(x,z) - T_\infty = T_0(x) + z(\zeta T)_{z=0} + (z^2/2!)(\zeta^2 T)_{z=0} + \cdots \quad (6)$$

where $\zeta = \partial(\ )/\partial z$ represents differentiation with respect to z. With the intermediate definition of

$$q(x, z) = -\partial T/\partial z$$

or

$$q = -\zeta T, \quad (7)$$

equation (1) takes on the form

$$\partial q/\partial z = \partial^2 T/\partial x^2 + 4Q\,e^{-2x}$$

or

$$\zeta q = \alpha^2 T + 4Q\,e^{-2x} \quad (8)$$

where $\alpha = \partial(\ )/\partial x$ represents differentiation with respect to x. It then follows from equations (7) and (8) that

$$\zeta T = -q, \quad \zeta^2 T = -\zeta q = -\alpha^2 T - 4Q\,e^{-2x}$$

$$\zeta^3 T = -\alpha^2 \zeta T = \alpha^2 q, \quad \zeta^4 T = \alpha^2 \zeta q = \alpha^4 T + 4Q2^2\,e^{-2x}$$

and so forth. Introducing these relationships into equation (6) gives

$$T(x,z) - T_\infty = \cos(z\alpha)T_0(x) - [\sin(z\alpha)/\alpha]q_0(x) + [\cos(2z) - 1]Q\,e^{-2x} \quad (9)$$

with the trigonometric functions compactly representing the series of derivatives involved. Requiring equation (9) to satisfy equations (3) and (4) leads to the relationships

$$-L_1 T_0 + L_2 q_0 = G$$

and

$$L_1 T_0 + L_2 q_0 = -G$$

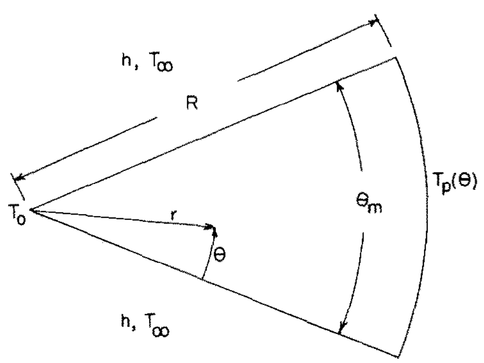

FIG. 1. Physical configuration and coordinate system.

where

$$L_1 = \alpha \sin(\theta_m \alpha/2) - B_i\,e^{-x}\cos(\theta_m \alpha/2)$$

$$L_2 = \cos(\theta_m \alpha/2) + B_i\,e^{-x}\sin(\theta_m \alpha/2)/\alpha$$

$$G = \{2\sin(\theta_m) + B_i[1 - \cos(\theta_m)]\,e^{-x}\}\,Q\,e^{-2x}.$$

These two relationships are satisfied by requiring that

$$L_1 T_0 = -G \quad (10)$$

and

$$L_2 q_0 = 0. \quad (11)$$

The solutions of equations (10) and (11) are assumed to be of the forms

$$T_0 = e^{-mx}(a_0 + a_1\,e^{-x} + a_2\,e^{-2x} + \cdots + a_n\,e^{-nx} + \cdots)$$

and

$$q_0 = e^{-mx}(b_0 + b_1\,e^{-x} + b_2\,e^{-2x} + \cdots + b_n\,e^{-nx} + \cdots)$$

which both satisfy equation (2). Substituting them into equations (10) and (11) and equating coefficients of like powers of e^{-x} allows the a_ns and b_ns to be evaluated.

For $\theta_m = \pi$ it is found that

$$T_0 = C_0[1 + B_i\,e^{-x}] + C_2[e^{-2x} + B_i\,e^{-3x}/3] + \cdots + C_n[e^{-nx} + B_i\,e^{-(n+1)x}/(n+1)] + \cdots + 2B_i Q\,e^{-3x}/3 \quad (12)$$

and

$$q_0 = C_1[e^{-x} + B_i\,e^{-2x}/2] + C_3[e^{-3x} + B_i\,e^{-4x}/4] + \cdots + C_{n+1}[e^{-(n+1)x} + B_i\,e^{-(n+2)x}/(n+2)] + \cdots \quad (13)$$

which, when substituted into equation (9), yield

$$T(x,z) - T_\infty = \sum_{n=0,2,4,\ldots}^{\infty} C_n\{\cos(nz)\,e^{-nx} + B_i \times \cos[(n+1)z]\,e^{-(n+1)x}/(n+1)\} + \sum_{n=1,3,5,\ldots}^{\infty} C_n\{\sin(nz)\,e^{-nx} + B_i \sin[(n+1)z]\,e^{-(n+1)x}/(n+1)\} - Q[e^{-2x} - (B_i/3)\cos(3z)\,e^{-3x})]. \quad (14)$$

The vertex temperature, the quantity of ultimate interest, is available from equation (14) if C_0 is known.

Equation (5) requires that

$$y(z) = \sum_{n=0}^{\infty} C_n f_n(z). \quad (15)$$

Since no weighting function with respect to which the $f_n(z)$ are orthogonal over the range $-\theta_m/2 \leqslant z \leqslant \theta_m/2$ was found, Rao's suggestion that equation (15) be satisfied in a least-squares sense as

$$E = \int_{-\theta_m/2}^{\theta_m/2} \left(y - \sum_{n=0}^{\infty} C_n f_n\right)^2 dz \rightarrow \text{minimum} \quad (16)$$

is used here for a finite number of $f_n(z)$. If the temperature on the circular arc is proportional to one of the f_n, an exact solution has a finite number of terms. One-dimensional conduction perpendicular to a straight boundary is such a case.

Retention of the first two terms in equations (14) and (16) leads to the requirement that $\partial E/\partial C_0 = 0 = \partial E/\partial C_1$ from which it is found that

$$C_0 = \int_{-\theta_m/2}^{\theta_m/2} y(z)\left[\frac{f_0(z)\overline{f_1^2} - f_1(z)\overline{f_1 f_0}}{\overline{f_0^2}\,\overline{f_1^2} - \overline{f_0 f_1}\,\overline{f_1 f_0}}\right] dz.$$

In general

$$C_0 = \int_{-\theta_m/2}^{\theta_m/2} y(z)P(z)\,\mathrm{d}z \qquad (17)$$

where the one-term form has

$$P(z) = \frac{1+B_i\cos(z)}{\pi+4B_i+\pi B_i^2/2} \qquad (18)$$

the two-term form has

$$P(z) = \frac{[1+B_i\cos(z)]-[\cos(2z)+(B_i/3)\cos(3z)]\,[(4B_i/9)/\overline{f_2^2}]}{[\pi+4B_i+\pi B_i^2/2]-[4B_i/9]\,[(4B_i/9)/\overline{f_2^2}]} \qquad (19)$$

and the three-term form has

$$P(z) = \frac{\begin{array}{r}[1+B_i\cos(z)]-[\cos(2z)+(B_i/3)\cos(3z)]\,\{[4^2B_i^2/21\cdot 75+(4B_i/9)\overline{f_4^2}]/[\overline{f_2^2}\,\overline{f_4^2}-(4B_i/21)^2]\}\\ +[\cos(4z)+(B_i/5)\cos(5z)]\,\{[4^2B_i^2/9\cdot 21+(4B_i/75)\,\overline{f_2^2}]/[\overline{f_2^2}\,\overline{f_4^4}-(4B_i/21)^2]\}\end{array}}{\begin{array}{r}[\pi+4B_i+\pi B_i^2/2]-[4B_i/9]\,\{[4^2B_i^2/21\cdot 75+(4B_i/9)\overline{f_4^2}]/[\overline{f_2^2}\,\overline{f_4^2}-(4B_i/21)^2]\}\\ -[4B_i/75]\,\{[4^2B_i^2/9\cdot 21+(4B_i/75)\,\overline{f_2^2}]/[\overline{f_2^2}\,\overline{f_4^2}-(4B_i/21)^2]\}.\end{array}} \qquad (20)$$

The vertex temperature for the semicircle is found by introducing one of these approximations for $P(z)$ into equation (17) to be

$$T_v - T_\infty = \int_{-\pi/2}^{\pi/2} [T_p(z)-T_\infty]P(z)\,\mathrm{d}z + Q\int_{-\pi/2}^{\pi/2} [1-(B_i/3)\cos(3z)]P(z)\,\mathrm{d}z. \qquad (21)$$

DISCUSSION AND MONTE CARLO APPLICATION

The number of terms needed for accuracy in $P(z)$ of equation (17) was determined by comparing the predictions of equations (18)–(20) with finite difference solutions. If errors of 1% or less are acceptable, the one-term form of equation (18) is satisfactory for $B_i < 1$ while the three-term form of equation (20) is satisfactory for all B_i.

Equation (21) for a semicircle can be applied to a Monte Carlo procedure at a straight convecting boundary, the most common boundary shape, by recasting it into the form

$$T_v = \int_0^A T_p\,\mathrm{d}F + \int_A^1 T_\infty\,\mathrm{d}F + QB. \qquad (22)$$

In a floating random walk Monte Carlo application, equation (22) is interpreted in the usual way [1]. The next move of a walker on a straight convecting boundary, the vertex of the semicircle considered in detail here, is determined by drawing a uniformly distributed random number N such that $0 \leqslant N \leqslant 1$. For $A \leqslant N \leqslant 1$, the known environmental temperature T_∞ is recorded and the random walk stops (the walker is absorbed). For $0 \leqslant N \leqslant A$, the walker moves into the interior (reflects) to the circular arc a distance R from the vertex at an angular position θ given by the solution to $F(\theta) = N$. In either case, the quantity QB is added to the walker's score. The radius R is the distance from the vertex, the position of the walker on the straight boundary for the semicircle discussed in detail here, to the nearest different straight-line boundary segment. Little generality is lost by considering the boundary to be made up of straight-line segments since this is often the actual situation and is a close approximation in many other cases. The behaviors of $F(\theta)$, a cumulative distribution function, and its derivative, a probability density function, vs. θ are shown in Fig. 2 for the cases of $B_i = 0$ and 4 for the one-term, two-term, and three-term forms of $P(z)$. When B_i is small all angles of reflection are equally likely, as would be expected for an insulated surface. As B_i increases, the direction perpendicular to the surface is increasingly favored as would be expected for one-dimensional conduction at an isothermal surface. A walker is allowed in a natural way to occasionally remain near a convecting boundary, rather than forcing it to always do so by the artificial means of a small step size.

The results of this study account for a possible two-dimensional temperature distribution near a straight convecting boundary and do not require perpendicular reflection. The advantage this confers can be estimated since the average distance reflected perpendicular to the straight convecting boundary in the suggested method is $d_{ave} \approx 2R/\pi$

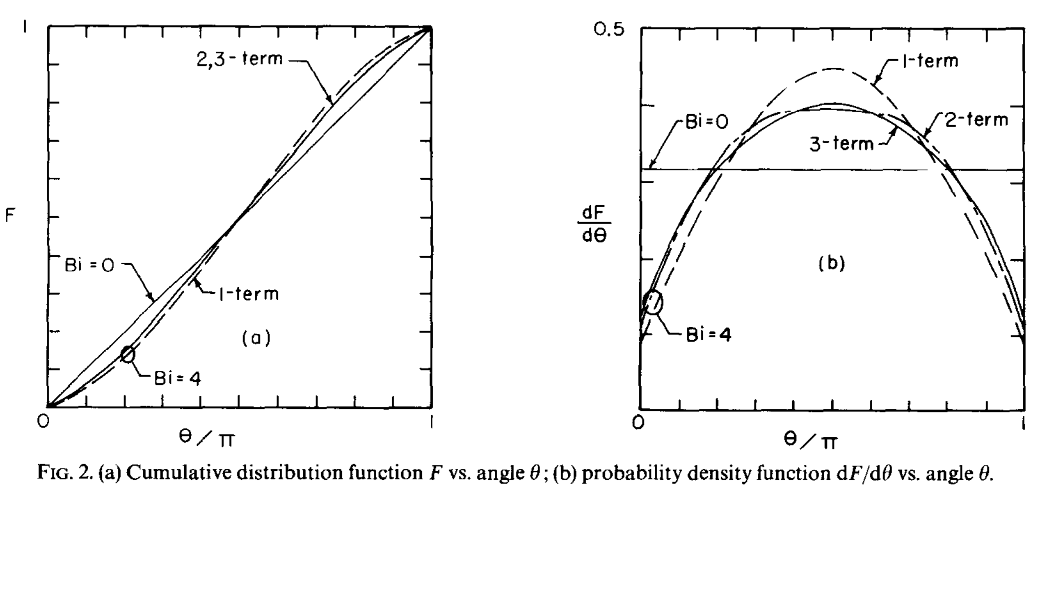

FIG. 2. (a) Cumulative distribution function F vs. angle θ; (b) probability density function $\mathrm{d}F/\mathrm{d}\theta$ vs. angle θ.

while the distance reflected in a one-dimensional finite difference approximation is limited by the criterion $hL/k \lesssim 1/20$ [3]. Since the present results allow $B_i = 4$ at least, $d_{ave}/L \approx 50$ or more. Because the time spent computing the distance from the position of a walker to the nearest boundary of a domain is the most time consuming part of a Monte Carlo procedure [4–6] this increase in average step size is significant since fewer steps would be required to complete a walk.

A less accurate single formulation for all included angles is available [8], obtained through application of the heat balance integral method, with less than about 1% error for $B_i < 1$.

Acknowledgement—The partial support of the National Science Foundation under Grant No. ENG77-27614 is acknowledged.

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Int. J. Heat Mass Transfer. Vol. 28, No. 3, pp. 720–723, 1985
Printed in Great Britain
0017-9310/85 $3.00 + 0.00
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A numerical study of natural convection in a vertical, annular, porous layer

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(*Received* 17 *January* 1984 *and in revised form* 5 *July* 1984)

1. INTRODUCTION

IN THIS note, we summarize the results of a numerical study of natural convection in an annular porous layer. This work was undertaken in an effort to gain insight into the heat loss mechanisms associated with large insulated tanks. The original presentation of this work, as a conference paper [1], preceeded the publication of more comprehensive studies of the same geometry by Havstad and Burns [2] and Prasad and Kulacki [3]. Prior to the appearance of these papers, analyses were available only for planar porous layers, for which refs [4–7] are representative. Given the recent interest shown in annular porous layers, it seems appropriate to consider briefly the main results of ref. [1]. A numerical method was used in this work that is different from the methods used in either of the other two studies cited. In addition, the effects of anisotropic permeability are given brief consideration. The present work is thus complementary to refs [2 and 3].

2. THE VERTICAL, ANNULAR, POROUS LAYER

The two-dimensional, axisymmetric configuration which is to be studied is illustrated in Fig. 1. The height of the layer is denoted by H, the thickness by W, and the inner and outer radii by R_i and R_0, respectively. All boundaries of the annular region are impermeable. Both horizontal boundaries are adiabatic and the vertical boundaries are maintained at constant temperatures T_i and T_0 where it is assumed that $T_i > T_0$. The annular region is filled with a rigid, fluid-saturated, porous medium. Gravity acts in the negative z-direction. In general, the permeability in the radial direction can differ from that associated with the vertical direction.

It is convenient to normalize all lengths with respect to the inner radius R_i. The annular region can then be represented by an equivalent annular region with an inner radius of unity, a height of H/R_i, and a thickness of W/R_i, where $W = R_0 - R_i$. Then, the annular region can be defined by specifying the layer aspect ratio H/W and the nondimensional height H/R_i. These latter two parameters are utilized in the presentation of the numerical results.

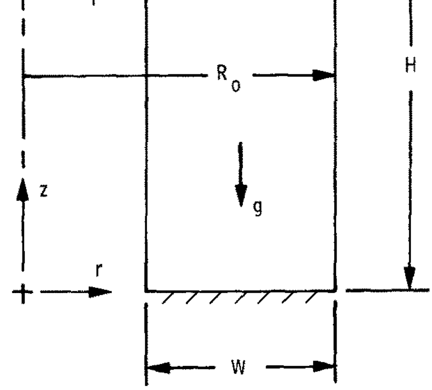

FIG. 1. The vertical, annular, porous layer.